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Inverse determination of the loading source of the infinite beam on elastic foundation †

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Abstract

The primary aim of the paper is to identify the loading source of infinite beams on an elastic foundation from given information of vertical deflection of infinite beams. An integral equation is obtained for the relationship between loading distribution and vertical deflection. It is shown that the inverse identification of a loading source is one-to-one but ill-posed. Because of ill-posedness, the usual numerical schemes produce arbitrarily large errors. A method for the solution is proposed by using Tikhonov's regularization. L-curve criterion is introduced for the determination of optimal regularization parameter. Numerical experiments show that the present methodology is accurate and robust in the inverse determination of loading source.

Keywords: Inverse problem; Integral equation; Ill-posed; Tikhonov's regularization; L-curve criterion

1. Introduction

Recently, inverse problems in the beam equation have been of practical interest, with examples including the identification of such material parameters as the distribution of Young's modulus or flexural rigidity as demonstrated in [1, 2]. Application areas can be measuring devices of micromechanics and flaw-identification by non-destructive methods, etc. Lucchinetti & Stüssi [3] formulated an inverse problem of the beam, *i.e.,* the identification of the distribution of the flexural rigidity of the beam by measuring its deflections. They showed that the problem is ill-posed in the sense of stability, and the ill-posedness leads to an unrealistic inverse solution of flexural rigidity when one treats the problem without any regularization procedure.

The present study aims to compute inversely the loading distribution from measured deflection of an infinite beam. This kind of approach can be found in [2] for identification of an unknown load applied to a steel-concrete composite beam by using measurements of the inclination along the axis of the beam, and in [4] for estimation of a heat source applied in the electron beam welding process by using temperature measurements in the solid phase.

Our motivation started from the offshore hydrodynamic viewpoint: the concept of very large floating structures [5-7] and ice plates in waves [8], which can be modeled with a beam equation when simplified to two-dimensional problems. One difference from the previous studies of inverse identification in the beam equation is the inclusion of an elastic foundation due to the buoyancy of the water, which is proportional to the local deflection. In this case, the relationship between the loading distribution and vertical deflection of the beam is expressed in the form of an integral equation of the first kind. It is assumed that the deflection is measured at a finite number of discrete

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points. Thus the inverse problem that identifies or recognizes the loading distribution of an infinite beam on an elastic foundation can be defined. It is shown that the concerning inverse problem is ill-posed in the sense of stability and one-to-one (identifiable).

Any direct numerical treatment of ill-posed problems (in the sense of stability) gives only a meaningless solution because conventional numerical approaches lead to arbitrarily large errors for the solution as indicated [9-11]. This phenomenon is due to the discontinuity of the problem (lack of stability). The ill-posedness can be effectively treated by regularization theory. In this study, the instability of ill-posedness is solved by introducing Tikhonov's regularization method. Regularization theory has been known as a very powerful mathematical tool to treat usual ill-posed inverse problems in the natural sciences and engineering [12-19].

This paper is organized as follows. The mathematical formulation for an infinite beam on an elastic foundation is reviewed in Section 2. An integral equation for the loading distribution is formulated and its uniqueness is discussed in Section 3. Stability is examined in Section 4. Tikhonov's regularization is discussed briefly in Section 5. Finally, numerical solutions are examined with the help of the regularization in Section 6.

2. Infinite beam on an elastic foundation

Let us consider an infinitely long beam on an elastic foundation, as depicted in Fig. 1. The foundation modulus of the spring stiffness per unit length is denoted as the constant k. In practical terms, the beam may be a train track with an elastic foundation that models the track bed.

From the classical Euler beam theory [20], the governing equation for the vertical deflection $u(x)$ that results from a load distribution of $w(x)$ (Newtons per unit length) is expressed by the fourth-order differential equation.

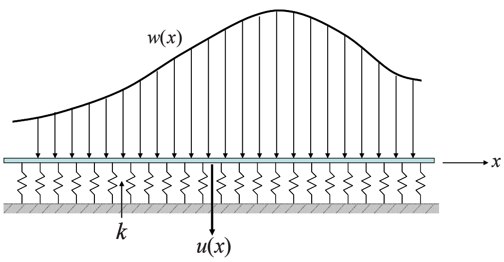


Fig. 1. Definition sketch for infinite beam on elastic foundation.

$$EI\frac{d^4u(x)}{dx^4} + ku(x) = w(x) \, , \tag{1}$$

where E is Young's modulus, and I the mass moment of inertia: EI is the flexural rigidity of the beam. Being divided by the flexural rigidity EI, the equation can be simplified as

$$\frac{d^4u(x)}{dx^4} + \alpha^4 u(x) = W(x) \, , \tag{2}$$

where

$$\alpha^4 = k / EI \quad \text{and} \quad W(x) = w(x)/EI \, . \tag{3}$$

Suppose that the loading $W(x)$ is localized enough so that u, du/dx, d^2u/dx^2, and d^3u/dx^3 all tend towards zero as $x \to \pm\infty$; then by using Fourier transform, the general solution of Eq. (1) is expressed in closed form in terms of a Green's function as follows:

$$u(x) = \int_{-\infty}^{\infty} w(\xi)G(\xi, x)d\xi \tag{4}$$

The Green's function is calculated by the complex contour integration to yield [21]:

$$G(\xi, x) = \frac{\alpha}{2k} e^{-\alpha|\xi - x|/\sqrt{2}} \sin\left(\frac{\alpha|\xi - x|}{\sqrt{2}} + \frac{\pi}{4} \right) . \tag{5}$$

3. Integral equation

In the study, finitely distributed loading is considered: in other words, a loading distribution on a finite interval. Let the loading distribution be denoted as $w(x)$, defined on a finite interval $(0, a)$, as shown in

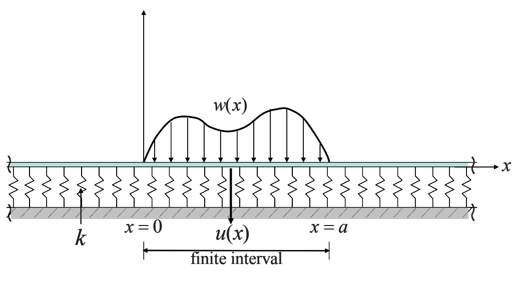


Fig. 2. Loading distribution on a finite interval.

Fig. 2, then the upper and lower limits in Eq. (4) are changed to a and 0, respectively:

$$u(x) = \int_0^a w(\xi) G(\xi, x) d\xi \,. \tag{6}$$

Eq. (6) can be rewritten in symbolic form or operator notation as follows:

$$u = \mathfrak{L}(w) \,, \tag{7}$$

where the operator $\mathfrak{L}$ is defined as:

$$\mathfrak{L}(w) = \int_0^a w(\xi) G(\xi, x) d\xi \,. \tag{8}$$

The focus of this study is an inverse problem that identifies the loading distributions of an infinite beam on an elastic foundation when the vertical deflection of the beam is known. Given the vertical deflection of the beam, the left side of Eq. (6) is known; thus, Eq. (6) becomes an integral equation in obtaining the unknown loading. The identification is realized by solving the integral Eq. (6). In the present inverse problem of identification, it is crucial to recover the real physical loading distribution of the infinite beam. Thus, it must be ascertained whether the problem has a unique solution so that the present problem is identifiable. We will examine the identifiability of the problem by showing uniqueness for the integral Eq. (6). We introduce a solution space X for the linear operator $\mathfrak{L}$ in Eq. (8) as a Hilbert space $H = L^2(0,a)$; that is,

$$X = H = \left\{ w : \int_0^a \left| w(\xi) \right|^2 d\xi < \infty \right\} . \tag{9}$$

In order to show that the operator $\mathfrak{L}$ is one-to-one, the null space $N(\mathfrak{L}) = \{x \in X : \mathfrak{L}x = o\}$ must be trivial [9]: that is, $N(\mathfrak{L}) = \{o\}$. This can be easily understood by physical consideration: no vertical deflection implies no loading in the present problem. Therefore, the present inverse problem to find the loading distribution has a unique solution.

4. Instability

Although the uniqueness of solution has been shown, the question of stability remains: that is, it should be verified whether the solution depends on the input data of given deflection of the beam or not.

The kernel of Green's function G in Eq. (6) is classified as a Hilbert-Schmidt kernel since the kernel is regular [21]. This makes the integral operator $\mathfrak{L}$ compact [12, 22]. Thus, Eq. (6) is a Fredholm integral equation of the first kind with a compact integral operator [12, 22]. Due to the compactness, the inverse $\mathfrak{L}^{-1}$ of the operator $\mathfrak{L}$ is discontinuous even though $\mathfrak{L}$ is continuous [9]. This implies that the integral Eq. (6) is ill-posed in the sense of stability.

The introduction of direct numerical methods into ill-posed problems only gives solutions with arbitrary large errors. In fact, a direct numerical discretization of the right side of Eq. (6) gives rise to a matrix whose condition number is very large enough that a numerical inverse of the matrix is not viable because the determinant of the matrix is almost zero: detailed explanation with numerical result is added in section 6. In general, ill-posed problems are known to cause numerical difficulties such as large errors.

5. Tikhonov regularization

Tikhonov [10] introduced a functional M which has a damping term Ω with a positive real number, β which is called the regularization parameter, to regularize the Fredholm integral equation of Eq. (6):

$$M = \left\{ \int_{-b}^{b} \left| \int_0^a w(\xi) G(\xi, x) d\xi - u(x) \right|^2 dx \right\}^{1/2} + \beta\,\Omega \,, \tag{10}$$

where b denotes the half length of measurement for the deflection $u(x)$: data for u is measured on the following finite interval:

$$\Gamma = (-b, b) \,. \tag{11}$$

The functional M in Eq. (10) is called the Tikhonov functional and the additional term Ω is defined , in this study, as follows:

$$\Omega = \int_0^a \left| w(\xi) \right|^2 \, d\xi \,. \tag{12}$$

Tikhonov functional M has a unique minimum w. This minimum is the unique solution of the normal equation, which is the Fredholm integral equation of second kind [9]:

$$\beta w + \mathfrak{L}^* \mathfrak{L} w = \mathfrak{L}^* u \,, \tag{13}$$

where $\mathfrak{L}^*$ is the adjoint operator, that is, for some function g,

$$\mathfrak{L}^* g = \int_0^a G(\xi, x) g(x) dx \,. \tag{14}$$

The loading distribution w can be determined by solving Eq. (13) with the measured vertical deflection u.

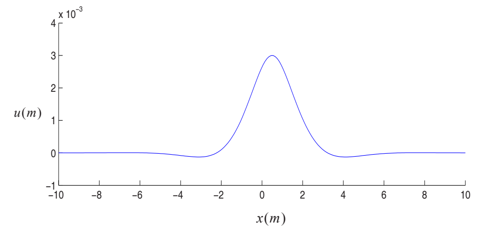
6. Numerical experiments

This section is devoted to numerical experiments that show the identification of loading source by using the regularization theory discussed in the previous section. Let us consider the following vertical deflection (Fig. 3(a)):

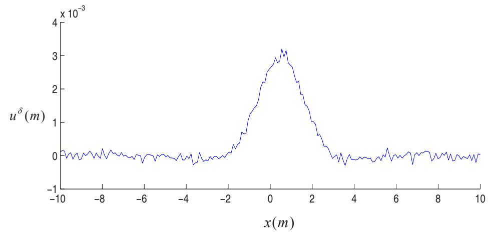
$$\begin{aligned} u(x) &= F\int_0^1 G(\xi, x) d\xi \\ &= \frac{\alpha F}{2k}\int_0^1 e^{-\alpha|\xi - x|/\sqrt{2}} \sin\left(\frac{\alpha|\xi - x|}{\sqrt{2}} + \frac{\pi}{4}\right) d\xi. \end{aligned} \tag{15}$$

The cause for vertical deflection in Eq. (15) is the following loading distribution for a constant F ($a = 1$):

$$w(x) = \begin{cases} F, & 0 < x < 1 \\ 0, & otherwise \end{cases} \,. \tag{16}$$



(a) u : Noise-free



(b) u^δ : Noise level $\delta = 0.0001$ (m)

Fig. 3. Vertical deflection of the beam in Eq. (15).

We begin with the vertical deflection of Eq. (15), which is considered as given data for the present numerical inverse study: that is, using the data information of the deflection, the corresponding loading distribution is to be determined. In practice, measured data are deteriorated to an extent by noise. We never know exactly the left hand-side in Eq. (6) but only up to an error of, say, noise level $\delta > 0$. In this study, we assume that we know δ and noisy data u^δ with

$$\left\| u - u^\delta \right\|_2 \le \delta \,, \tag{17}$$

where the symbol $\|\cdot\|_2$ means L_2 norm [9]. Now it is our aim to solve the perturbed equation. For the numerical experiments, we choose an error intensity $\delta = 0.0001$ (m) and generate noisy data randomly. The randomly generated deflection is illustrated in Fig. 3(b).

In the first place, without the aid of any regularization, we discretize the integral equation in Eq. (6) with the trapezoidal rule, to find a direct numerical solution to Eq. (6) as shown in Fig. 4 (the number of discretizations is chosen as 100). Here, the physical coefficients k (foundation modulus), I (moment of inertia), E (Young's modulus), and F are assumed to be $14\ \mathrm{MPa}$, $2.8125\times10^{-5}\ m^4$, $200\ \mathrm{GPa}$, and $100\ \mathrm{kN/m}$, respectively. All the obtained solutions are found to be meaningless. In fact, the behavior of the solutions has no regularity at all. Furthermore, we note that their magnitudes are unrealistically high. In order to analyze the source of the trouble, it is worth checking the condition number for the discretized form of the operator $\mathfrak{L}$ in Eq. (8). The result in Fig. 5 shows that most of the condition numbers have extremely large values, which implies that the present numerical system for the direct discretization is ill-conditioned. That is why we have the unstable meaningless solutions as shown in Fig. 4.

We have shown that direct numerical treatment fails to solve the present inverse problem. Therefore, as sketched in section 5, Tikhonov's regularization theory is to be applied to solve the inverse problem. According to the regularization theory, the regularization parameter β in Eq. (13) plays an important role in the method of Tikhonov's regularization.

Now the problem is how to select the regularization parameter to obtain the optimal solution. In this study, the L-curve criterion [23] is introduced to determine the appropriate regularization parameter.

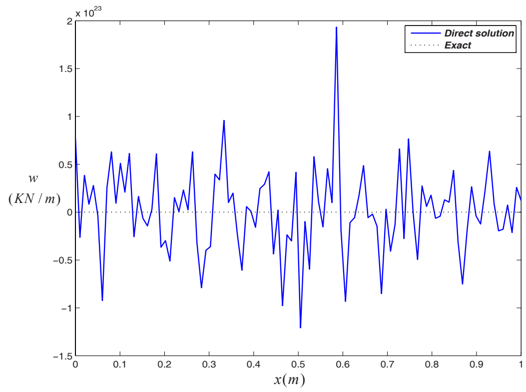


Fig. 4. Numerical solution of direct discretization (typical example of instability).

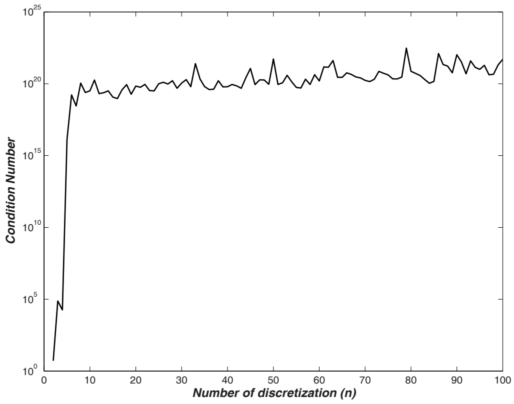


Fig. 5. Distribution of condition number for the discretized operator $\mathfrak{L}$ in Eq. (8).

Considerable computational experience indicates that the L-curve criterion is a powerful method for determining a suitable value of the regularization parameter for many problems of interest in science and engineering.

The L-curve is represented as a log-log plot of the norm of a regularized solution versus the norm of the corresponding residual as the regularization parameter is varied:

$$\left(\log\left\|\mathfrak{L}w-u^{\delta}\right\|_2, \quad \log\left\|w\right\|_2\right). \tag{18}$$

This curve exhibits a typical "L" shape, and the optimal value for the regularization parameter is considered to be the one that corresponds to the corner of the curve (Hansen 1992). A heuristic motivation for this choice is that when the regularization parameter

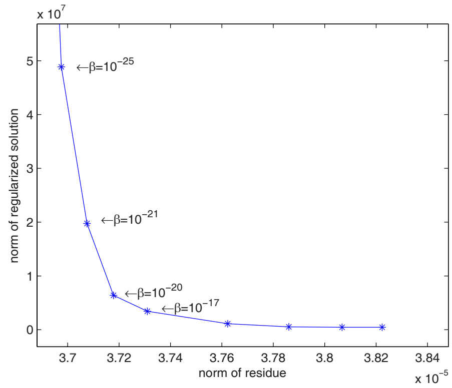
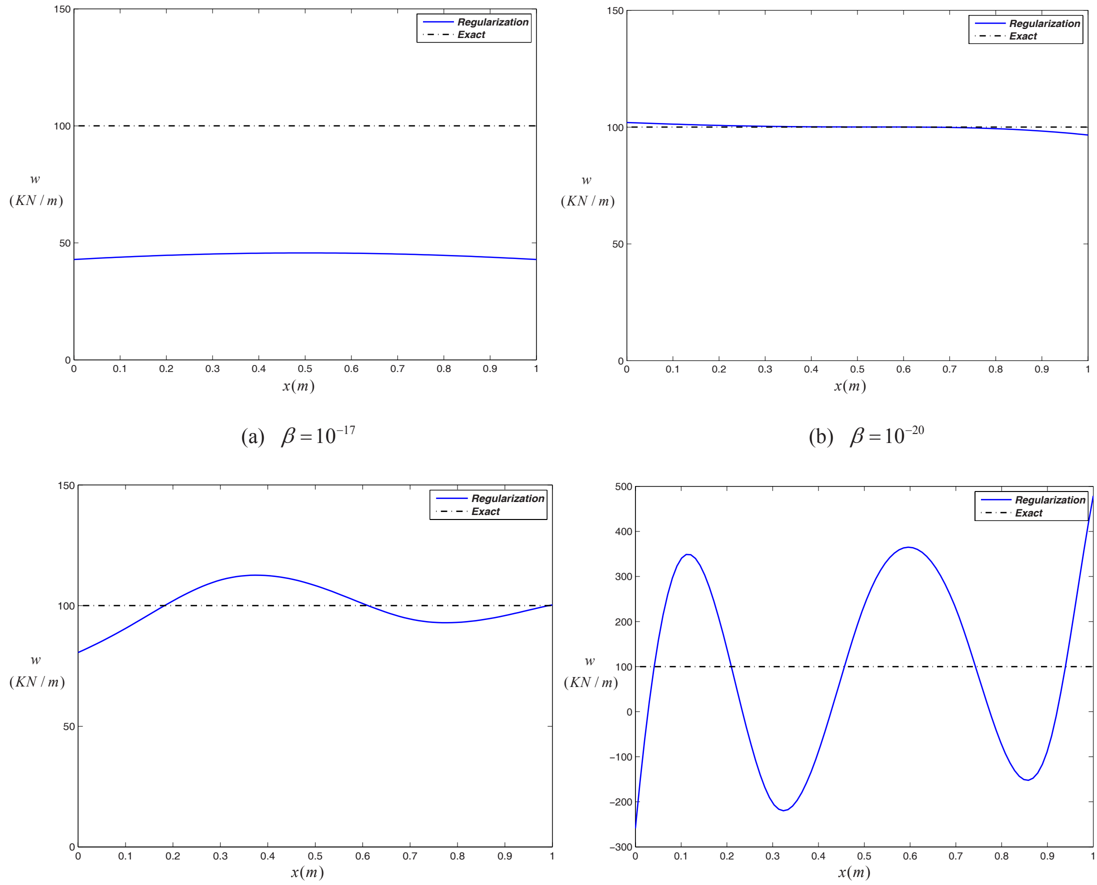


Fig. 6. Graphical illustration of L-curve.

is small, then the norm of associated solution w is huge and at the same time it is likely to be contaminated by measurement errors. Conversely, when the regularization parameter is large, the solution w is a poor approximation and the norm of associated difference $\left\|\mathfrak{L}w-u^{\delta}\right\|_2$ is huge. The corner of the L-curve marks this transition, since it represents a compromise between the minimization of the norm of the residual $\left\|\mathfrak{L}w-u^{\delta}\right\|_2$ and norm of the solution $\|w\|_2$. This choice of the regularization parameter of Tikhonov's regularization is not guaranteed to be appropriate for all linear systems with a very ill-conditioned system. However, considerable computational experience indicates that the L-curve criterion is a powerful method for determining a suitable value of the regularization parameter for many problems of interest in science and engineering. Fig. 6 shows the graphical illustration for the present numerical experiment. It should be pointed out that the L-curve criterion, combined with Tikhonov's regularization, makes it possible to obtain numerical solutions even though we do not know the exact solution.

We choose the value of the corner of L-curve as the appropriate regularization parameter($\beta = 10^{-20}$) and compute Tikhonov's regularization to obtain the loading distribution w using the given noisy vertical deflection u^{δ}. The regularized solution is depicted in Fig. 7. The figure clearly shows that the case of the optimal regularization parameter $\beta = 10^{-20}$ gives the most accurate result compared with other cases of regularization parameters. In contrast to the unstable results (without regularization) in Fig. 4, a fairly stable solution is obtained, which is accurate when compared to the exact solution in Eq. (16).



(a) $\beta = 10^{-17}$ (b) $\beta = 10^{-20}$

(c) $\beta = 10^{-21}$ (d) $\beta = 10^{-25}$

Fig. 7. Regularized solutions (with Tikhonov regularization): In the case of (d), note that a different vertical scale is used.

7. Summary and conclusions

This paper has dealt with the inverse problem of identification of loading source from the measured data of vertical deflection. In this work, it has been newly proved that the identification of loading source is ill-posed in the sense of stability. That is, the solution does not depend continuously on the data of measured vertical deflection. Because of the ill-posedness of the problem, it was seen that a small measurement error of data would give an arbitrarily large error in conventional methods of solution. This means that it is impossible to recover the loading distribution with usual numerical schemes. Thus an efficient methodology was proposed by using Tikhonov's regularization. For realistic data acquisition, the present study introduced a concept of noise level to the problem. Optimal regularization parameter depending upon noise level was investigated with the help of the L-curve criterion. The numerical experiments showed that the present method of inversion is accurate and robust in inverse determination of the information about the loading source with the measured data of vertical deflection.

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